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Convergence to diffusion waves for nonlinear evolution equations with different end states ☆

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Abstract

In this paper, we consider the global existence and the asymptotic decay of solutions to the Cauchy problem for the following nonlinear evolution equations with ellipticity and dissipative effects:

$$\begin{cases} \psi_t = -(1-\alpha)\psi + \psi\psi_x + \big(f(\theta)\big)_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\alpha)\theta + \nu\psi_x + (\psi\theta)_x + \alpha\theta_{xx}, \end{cases} \tag{E}$$

with initial data

$$(\psi,\theta)(x,0) = \big(\psi_0(x),\theta_0(x)\big) \to (\psi_\pm,\theta_\pm) \quad \text{as } x \to \pm\infty, \tag{I}$$

where α and ν are positive constants such that $\alpha < 1$, $s\nu < 4\alpha(1-\alpha)$ (s is defined in (1.14)). Under the assumption that $|\psi_+ - \psi_-| + |\theta_+ - \theta_-|$ is sufficiently small, we show that if the initial data is a small perturbation of the diffusion waves defined by (2.5) which are obtained by the diffusion equations (2.1), solutions to Cauchy problem (E) and (I) tend asymptotically to those diffusion waves with exponential rates. The analysis is based on the energy method. The similar problem was studied by Tang and Zhao [S.Q. Tang, H.J. Zhao, Nonlinear stability for dissipative nonlinear evolution equations with ellipticity, J. Math. Anal. Appl. 233 (1999) 336–358] for the case of $(\psi_\pm,\theta_\pm) = (0,0)$.

Keywords: Evolution equations; Diffusion waves; Decay rate; Energy method; A priori estimates

1. Introduction

In paper [10], Jian, Wang, and Hsieh studied the dissipative nonlinear evolution equations with divergence form

$$\begin{cases} \psi_t = \alpha\psi + \lambda\psi\psi_x + \big(f(\theta)\big)_x + \varepsilon_1\psi_{xx}, \\ \theta_t = \beta\theta + \nu\psi_x + (\psi\theta)_x + \varepsilon_2\theta_{xx}, \end{cases} \quad 0 < x < L,\ t > 0, \tag{1.1}$$

with initial data

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$$\big(\psi(x,0),\theta(x,0)\big)=\big(\psi_0(x),\theta_0(x)\big), \tag{1.2}$$

and boundary condition

$$(\psi,\theta)(x,t)=(0,0), \quad x\in\{0,L\},\ t\geqslant 0, \tag{1.3}$$

where L, ε_1, and ε_2 are all constants, while α, β, λ, and ν are given real constants. The nonlinear term $f\in C^\infty_{\mathrm{loc}}(R)$ satisfies $|f_z(z)|\leqslant k$, $\forall z\in R$. Global smooth solution $(\psi,\theta)\in C([0,\infty), H_0^1([0,L])\cap C^\infty((0,L))\times(0,\infty)$ and the global attractor for the above dissipative nonlinear system is studied in the abstract theory method on evolution equations as in Henry [4].

There are also papers about related model studied by Hsiao and Jian in [5] and Jian and Chen in [9],

$$\begin{cases}\psi_t=-(\sigma-\alpha)\psi-\sigma\theta_x+\alpha\psi_{xx},\\ \theta_t=-(1-\beta)\theta+\nu\psi_x+(\psi\theta)_x+\beta\theta_{xx},\end{cases} \tag{1.4}$$

where α, β, σ and ν are all positive constants satisfying the relations $\alpha<\sigma$ and $\beta<1$. We refer to Hsieh [7] and Tang [15] for the physical background of (1.4).

If we ignore the damping and diffusion terms temporarily, system (1.1) is simplified as

$$\begin{pmatrix}\psi\\ \theta\end{pmatrix}_t=\begin{pmatrix}0 & -\sigma\\ \nu+\theta & \psi\end{pmatrix}\begin{pmatrix}\psi\\ \theta\end{pmatrix}_x. \tag{1.5}$$

It is easy to get the two characteristic values of the system (1.3) are

$$\lambda_\pm=\frac{1}{2}\big(\psi\pm\sqrt{\psi^2-4\sigma(\nu+\theta)}\big).$$

This means the system (1.5) is elliptic for $|\psi|<2\sqrt{\sigma(\nu+\theta)}$, and hyperbolic for $|\psi|>2\sqrt{\sigma(\nu+\theta)}$. Around the zero equilibrium, system (1.5) subject to initial small disturbance is unstable owing to the ellipticity and $|\psi|$ will grow since the inherent instability of system (1.5). When the growth of $|\psi|$ leads to $|\psi|>\sqrt{\sigma\nu}$, system (1.5) becomes hyperbolic at once and ψ ceases to grow. Thus, a "switching back and forth" phenomenon is expected due to the interplaying among ellipticity, hyperbolicity and dissipation for suitable coefficients, which makes system (1.5) quite complicated, even occur chaos. But we may predict the damping and diffusion terms' joining to system (1.5) will prevent ψ from growing and make the solutions stable.

As to the study of system (1.1), there are only a few rigorous results available so far due to the complexity of system (1.1) as we mentioned above, cf. [5,9].

In paper [5], Hsiao and Jian discussed the initial boundary value problem of (1.4) with initial data

$$\big(\psi(x,0),\theta(x,0)\big)=\big(\psi_0(x),\theta_0(x)\big), \tag{1.6}$$

and boundary condition

$$(\psi,\theta)(0,t)=(\psi,\theta)(1,t), \quad (\psi_x,\theta_x)(0,t)=(\psi_x,\theta_x)(1,t), \quad 0\leqslant t\leqslant T. \tag{1.7}$$

When initial data $(\psi_0(x),\theta_0(x))\in C^{2,\delta}([0,1])\times C^{2,\delta}([0,1])$, $0<\delta<1$, they established the global existence and uniqueness of classical solutions for the corresponding problem by applying the energy method and Leray–Schauder fixed point theorem.

In paper [9], Jian and Chen considered the Cauchy problem for system (1.4) with the initial data (1.6). When initial data

$$\big(\psi_0(x),\theta_0(x)\big)\in H^1(\mathbb{R})\cap L^1(\mathbb{R}), \tag{1.8}$$

they used the abstract theory concerning the initial value problem in a Banach space to obtain the existence and uniqueness of local solutions to (1.4) and (1.7). Then, by a priori estimates of H^1-norm, the global existence results were obtained.

On the other hand, the system with the same complicated properties was also studied by Tang and Zhao [16]. Precisely, Tang and Zhao considered the following Cauchy problem that was proposed by Hsieh [7]:

$$\begin{cases}\psi_t=-(1-\alpha)\psi-\theta_x+\alpha\psi_{xx},\\ \theta_t=-(1-\alpha)\theta+\nu\psi_x+2\psi\theta_x+\alpha\theta_{xx},\end{cases} \tag{1.9}$$

with initial data (1.6).

Under the assumption $\nu < 4\alpha(1-\alpha)$ and the initial data

$$\big(\psi_0(x), \theta_0(x)\big) \in L^2 \cap W^{1,\infty}\big(\mathbb{R}, \mathbb{R}^2\big), \tag{1.10}$$

they proved the global existence of the solutions to Cauchy problem (1.6), (1.4) and obtained the decay rates of the solutions by the Fourier analysis and the energy method. However, the assumption (1.6) in [9] or (1.10) in [16] implies

$$\big(\psi_0(x), \theta_0(x)\big) \to (0,0), \quad \text{as } x \to \pm\infty, \tag{1.11}$$

which is a rigorous restriction on the initial data $(\psi_0(x), \theta_0(x))$.

Very recently, Zhu and Wang [20], Duan and Zhu [2] extended Tang and Zhao's result to the case of more general initial data

$$\big(\psi(x,0), \theta(x,0)\big) = \big(\psi_0(x), \theta_0(x)\big) \to (\psi_\pm, \theta_\pm), \quad \text{as } x \to \pm\infty, \tag{1.12}$$

where $\psi_\pm$, $\theta_\pm$ are constant states and $(\psi_+ - \psi_-, \theta_+ - \theta_-) \neq (0,0)$. And the existence and the decay rates of the solutions of the corresponding problem was also obtained. For other results on this direction refer to [8,11,12,15].

Based on the idea in [2,20,21], we will consider the existence and asymptotic behavior of the solutions to Cauchy problem of (1.1). To make the analysis easier for reading, we take simple coefficients in (1.1). Now we consider the Cauchy problem

$$\begin{cases} \psi_t = -(1-\alpha)\psi + \psi\psi_x + \big(f(\theta)\big)_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\alpha)\theta + \nu\psi_x + (\psi\theta)_x + \alpha\theta_{xx}, \end{cases} \tag{1.13}$$

with initial data (1.12). The nonlinear term $f \in C^\infty_{\mathrm{loc}}(R)$ satisfies

$$\big|f^n(z)\big| \leqslant s \leqslant 1, \quad \forall z \in R,\ n = 1,2,3. \tag{1.14}$$

To establish the global existence and get the decay rates of the corresponding solutions by applying the energy method, our plan is arranged as follows:

First of all, we need to find the asymptotic profile, linear diffusion waves, defined by the linear diffusion equations, which are obtained by approximating the system (1.13), cf. (2.1). This will be done in Section 2. Secondly, in Section 3, the global existence and asymptotic behavior are obtained from the local existence and a priori estimates. Finally, in Section 4, we get decay rates to the diffusion waves obtained in Section 2 for the solutions to (1.13) and (1.12).

Notations. Throughout this paper, we denote positive constants by C. Moreover, the character "C" may differ in different places. $L^p = L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ denotes the usual Lebesgue space on $\mathbb{R} = (-\infty, \infty)$ with its norm $\|f\|_{L^p} = (\int_{\mathbb{R}} |f(x)|^p\,dx)^{\frac{1}{p}}$, $1 \leqslant p < \infty$, $\|f\|_{L^\infty} = \sup_{\mathbb{R}} |f(x)|$, and when $p = 2$, we write $\|\cdot\|_{L^2(\mathbb{R})} = \|\cdot\|$. $H^l(\mathbb{R})$ denotes the usual lth order Sobolev space with its norm $\|f\|_{H^l(\mathbb{R})} = \|f\|_l = (\sum_{i=0}^l \|\partial_x^i f\|^2)^{\frac{1}{2}}$. For simplicity, $\|f(\cdot,t)\|_{L^p}$ and $\|f(\cdot,t)\|_l$ are denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_l$, respectively.

2. Analysis of the linear diffusion waves

As in [2,6,14,19], we expect the solutions of (1.13) time-asymptotically behave as those of the following linear system:

$$\begin{cases} \bar\psi_t = -(1-\alpha)\bar\psi + \alpha\bar\psi_{xx}, \\ \bar\theta_t = -(1-\alpha)\bar\theta + \alpha\bar\theta_{xx}. \end{cases} \tag{2.1}$$

Since both equations in (2.1) are independent, it suffices to solve $(2.1)_1$. By setting the transformation $\bar\psi(x,t) = \bar\phi(x,t)e^{-(1-\alpha)t}$, we can derive a heat equation from $(2.1)_1$, that is

$$\bar\phi_t = \alpha\bar\phi_{xx}. \tag{2.2}$$

We hope to find the solution $\bar\phi(x,t)$ of the following form:

$$p(\xi) = p\left(\frac{x}{\sqrt{1+t}}\right), \quad -\infty < \xi < \infty, \tag{2.3}$$

satisfying boundary conditions $p(\pm\infty) = \psi_\pm$, where $\xi = \frac{x}{\sqrt{1+t}}$.

It follows from (2.2) and (2.3)

$$\begin{cases} -\dfrac{1}{2}\xi p'(\xi) = \alpha p''(\xi), \\ p(\pm\infty) = \psi_\pm. \end{cases} \tag{2.4}$$

We get by the direct calculation

$$\bar{\phi}(x,t) = p(\xi) = \frac{\psi_+ - \psi_-}{\sqrt{4\pi\alpha(1+t)}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{4\alpha(1+t)}\right) dy + \psi_-,$$

which gives the solutions of (2.1)

$$\begin{cases} \bar{\psi}(x,t) = e^{-(1-\alpha)t}\left((\psi_+ - \psi_-)\displaystyle\int_{-\infty}^{x} G(y,t+1)\,dy + \psi_-\right), \\ \bar{\theta}(x,t) = e^{-(1-\alpha)t}\left((\theta_+ - \theta_-)\displaystyle\int_{-\infty}^{x} G(y,t+1)\,dy + \theta_-\right), \end{cases} \tag{2.5}$$

where $G(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \exp(-\frac{x^2}{4\alpha t})$ is the heat kernel function. It is easy to show

$$\begin{cases} \bar{\psi}(x,t) \to \psi_\pm e^{-(1-\alpha)t}, & x \to \pm\infty, \\ \bar{\theta}(x,t) \to \theta_\pm e^{-(1-\alpha)t}, & x \to \pm\infty. \end{cases} \tag{2.6}$$

Now we will consider the asymptotic behavior of $\bar{\psi}(x,t)$, $\bar{\theta}(x,t)$ and their derivatives in $L^p(\mathbb{R})$. First for the heat kernel function, it has the following properties.

Lemma 2.1. *When $1 \leqslant p \leqslant +\infty$, $0 \leqslant l, k < +\infty$, we have*

$$\left\|\partial_t^l \partial_x^k G(x,t)\right\|_{L^p} \leqslant C t^{-\frac{1}{2}(1-\frac{1}{p})-l-\frac{k}{2}}.$$

From the above lemma, we can get the following results by simple calculations.

Lemma 2.2. *The solutions $\bar{\psi}(x,t)$ and $\bar{\theta}(x,t)$ to* (2.1) *satisfy the following properties*:

(i) $\|\partial_t^l \bar{\psi}(t)\|_{L^\infty} \leqslant Ce^{-(1-\alpha)t}$, $\|\partial_t^l \bar{\theta}(t)\|_{L^\infty} \leqslant Ce^{-(1-\alpha)t}$, $l = 0, 1, 2, \ldots$;
(ii) *for any p with $1 \leqslant p \leqslant +\infty$, it holds that*

$$\left\|\partial_t^l \partial_x^k \bar{\psi}(t)\right\|_{L^p} \leqslant C|\psi_+ - \psi_-| e^{-(1-\alpha)t}(1+t)^{\frac{1}{2p}-\frac{k}{2}}, \quad k = 1, 2, \ldots,\ l = 0, 1, 2, \ldots,$$
$$\left\|\partial_t^l \partial_x^k \bar{\theta}(t)\right\|_{L^p} \leqslant C|\theta_+ - \theta_-| e^{-(1-\alpha)t}(1+t)^{\frac{1}{2p}-\frac{k}{2}}, \quad k = 1, 2, \ldots,\ l = 0, 1, 2, \ldots.$$

3. Global existence and asymptotic behavior

3.1. Reformulation of the problem

Let

$$\begin{cases} u(x,t) = \psi(x,t) - \bar{\psi}(x,t), \\ v(x,t) = \theta(x,t) - \bar{\theta}(x,t). \end{cases} \tag{3.1}$$

Then from (2.1), we can rewrite problem (1.3) and (1.7) as follows

$$\begin{cases} u_t = -(1-\alpha)u + \alpha u_{xx} + uu_x + (u\bar{\psi})_x + f_\theta(v_x + \bar{\theta}_x) + E(x,t), \\ v_t = -(1-\alpha)v + \alpha v_{xx} + \nu u_x + (uv)_x + (\bar{\theta}u)_x + (\bar{\psi}v)_x + F(x,t), \end{cases} \tag{3.2}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x) = \psi_0(x) - \bar{\psi}(x,0) \to 0, & x \to \pm\infty, \\ v(x,0) = v_0(x) = \theta_0(x) - \bar{\theta}(x,0) \to 0, & x \to \pm\infty, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} &E(x,t) = \bar{\psi}\bar{\psi}_x, \\ &F(x,t) = \nu\bar{\psi}_x + \bar{\psi}\bar{\theta}_x + \bar{\theta}\bar{\psi}_x. \end{aligned} \tag{3.4}$$

We seek the solutions of (3.2), (3.3) in the set of function $X(0,T)$ defined by

$$X(0,T) = \{(u,v) \mid u, v \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3)\}.$$

Now we state our first main results as follows.

Theorem 3.1. *Let $(u_0(x), v_0(x)) \in H^2(\mathbb{R}, \mathbb{R}^2)$. Furthermore, suppose that both $\delta = |\psi_+ - \psi_-| + |\theta_+ - \theta_-|$ and $\delta_0 = \|u_0\|_2^2 + \|v_0\|_2^2$ are sufficiently small. Then for any $0 < \alpha < 1$, $\nu < 4\alpha(1-\alpha)$, the Cauchy problem* (3.2), (3.3) *admits a unique global solution $(u(x,t), v(x,t)) \in X(0,T)$ satisfying*

$$\|u(t)\|_2^2 + \|v(t)\|_2^2 + \int_0^t \left(\|u(\tau)\|_3^2 + \|v(\tau)\|_3^2\right) d\tau \leqslant C(\delta + \delta_0) \tag{3.5}$$

and

$$\sup_{x\in\mathbb{R}}\left(\left|(u,v)(x,t)\right| + \left|(u_x,v_x)(x,t)\right|\right) \to 0, \quad \text{as } t \to \infty. \tag{3.6}$$

3.2. Local existence

In this subsection, we will study the local existence of Cauchy problem (3.2) and (3.3). To this end, we rewrite the Cauchy problem (3.2) and (3.3) in the following integral forms:

$$\begin{cases} u(x,t) = G(x,t) * u_0(x) - (1-\alpha)\displaystyle\int_0^t G(x,t-s) * u(x,s)\,ds \\ \qquad - \displaystyle\int_0^t G_x(x,t-s) * \left(\frac{u^2}{2}\right)(x,s)\,ds - \displaystyle\int_0^t G_x(x,t-s) * (u\bar{\psi})(x,s)\,ds \\ \qquad + \displaystyle\int_0^t G(x,t-s) * (\bar{\psi}\bar{\psi}_x)(x,s)\,ds + \displaystyle\int_0^t G(x,t-s) * f_x(v+\bar{\theta})(x,s)\,ds, \\ v(x,t) = G(x,t) * v_0(x) - (1-\alpha)\displaystyle\int_0^t G(x,t-s) * v(x,s)\,ds \\ \qquad - \nu\displaystyle\int_0^t G_x(x,t-s) * u(x,s)\,ds - \displaystyle\int_0^t G_x(x,t-s) * (uv)(x,s)\,ds \\ \qquad - \displaystyle\int_0^t G_x(x,t-s) * (\bar{\theta}u)(x,s)\,ds - \displaystyle\int_0^t G_x(x,t-s) * (\bar{\psi}v)(x,s)\,ds + \displaystyle\int_0^t G(x,t-s) * F(x,s)\,ds, \end{cases}$$

where the convolutions are taken with respect to the space variable x. We can construct the approximate solution sequences and obtain the local existence by implementing the standard arguments with Brower fixed point principle [1].

Lemma 3.2 *(Local existence). If $(u_0(x), v_0(x)) \in H^2(\mathbb{R}, \mathbb{R}^2)$, then there exists t_0 depending only on $\|(u_0(x), v_0(x))\|_{H^2(\mathbb{R},\mathbb{R}^2)}$, such that the Cauchy problem* (3.2), (3.3) *admits a unique smooth solution $(u(x,t), v(x,t)) \in X(0, t_0)$ satisfying*

$$\big\|\big(u(x,t), v(x,t)\big)\big\|_{H^2(\mathbb{R},\mathbb{R}^2)} \leqslant 2\big\|\big(u_0(x), v_0(x)\big)\big\|_{H^2(\mathbb{R},\mathbb{R}^2)}. \tag{3.7}$$

3.3. Global existence and asymptotic behavior

By the local existence result, in order to get the global existence of Cauchy problem (3.2) and (3.3), it is sufficient to get a priori estimates. Precisely, we need to prove that there exists a constant C depending only on $\|(u_0(x), v_0(x))\|_{H^2(\mathbb{R},\mathbb{R}^2)}$ such that any solution $(u, v)(x, t)$ in $X(0, T)$ satisfy $\|u(t)\|_2 + \|v(t)\|_2 \leqslant C$ for any t in $[0, T]$. Next, we devote ourselves to the estimate of the solution $(u(x,t), v(x,t))$ of (3.2), (3.3) under the a priori assumption

$$N(T) = \sup_{0<t<T}\left\{\sum_{k=0}^{2}\big\|\partial_x^k u(t)\big\|^2 + \sum_{k=0}^{2}\big\|\partial_x^k v(t)\big\|^2\right\} \leqslant \delta_1^2, \tag{3.8}$$

where $0 < \delta_1 \ll 1$.

By Sobolev inequality $\|f\|_{L^\infty} \leqslant \|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}$, we have

$$\big\|(u, u_x, v, v_x)\big\|_{L^\infty} \leqslant \delta_1, \tag{3.9}$$

which will be used later.

Moreover, if $s\nu < 4\alpha(1-\alpha)$, we can find $\varepsilon \in (0, 2)$, $c_0 > 0$ such that

$$\begin{cases} 2c_0\alpha - \dfrac{s^2}{(1-\alpha)\varepsilon} > 0, \\ 2(1-\alpha) - \dfrac{c_0\nu^2}{\alpha\varepsilon} > 0. \end{cases} \tag{3.10}$$

In fact, from $s\nu < 4\alpha(1-\alpha)$, we know that there exists a constant $k \in (0, 1)$, such that $s\nu = 4k\alpha(1-\alpha)$. Choosing $\varepsilon = k + 1$ and $c_0 = \frac{s^2}{2\alpha(1-\alpha)}(\frac{1}{2(k+1)} + \frac{k+1}{8k^2})$, one can easily verify that ε and c_0 satisfy (3.10).

What follows will be a series of lemmas contributing to our desired estimates.

Lemma 3.3. *Suppose that the assumptions in Theorem* 3.1 *hold and $(u(x,t), v(x,t))$ is a solution to* (3.2), (3.3) *obtained in Lemma* 3.2, *then it holds that for any $\nu < 4\alpha(1-\alpha)$,*

$$\int_{\mathbb{R}}\big(u^2+v^2\big)\,dx + \int_0^t\int_{\mathbb{R}}\big(u^2+v^2\big)\,dx\,d\tau + \int_0^t\int_{\mathbb{R}}\big(u_x^2+v_x^2\big)\,dx\,d\tau \leqslant C(\delta+\delta_0), \tag{3.11}$$

provided δ and δ_1 are sufficiently small.

Proof. Multiplying the first equation of (3.2) by $2u$ and the second equation of (3.2) by $2c_0v$ and integrating the resulting identity over $\mathbb{R} \times (0, t)$, we arrive at by Cauchy–Schwartz inequality

$$\begin{aligned}
&\int_{\mathbb{R}}\big(u^2+c_0v^2\big)\,dx + 2(1-\alpha)\int_0^t\int_{\mathbb{R}}\big(u^2+c_0v^2\big)\,dx\,d\tau + 2\alpha\int_0^t\int_{\mathbb{R}}\big(u_x^2+c_0v_x^2\big)\,dx\,d\tau \\
&\quad = \|u_0\|^2 + c_0\|v_0\|^2 + 2\int_0^t\int_{\mathbb{R}} f_\theta u v_x\,dx\,d\tau + 2\nu c_0\int_0^t\int_{\mathbb{R}} v u_x\,dx\,d\tau - 2\int_0^t\int_{\mathbb{R}} f_\theta u\bar{\theta}_x\,dx\,d\tau \\
&\qquad + c_0\int_0^t\int_{\mathbb{R}} u_x v^2\,dx\,d\tau + c_0\int_0^t\int_{\mathbb{R}} \bar{\psi}_x v^2\,dx\,d\tau - 2c_0\int_0^t\int_{\mathbb{R}} \bar{\theta} u v_x\,dx\,d\tau + 2c_0\int_0^t\int_{\mathbb{R}} vF(x,\tau)\,dx\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}} u_x u^2\,dx\,d\tau+\int_0^t\int_{\mathbb{R}} \bar{\psi}_x u^2\,dx\,d\tau+2\int_0^t\int_{\mathbb{R}} uE(x,\tau)\,dx\,d\tau\\
&\leqslant (1+c_0)\delta_0+\varepsilon(1-\alpha)\int_0^t\int_{\mathbb{R}} u^2\,dx\,d\tau+\frac{s^2}{\varepsilon(1-\alpha)}\int_0^t\int_{\mathbb{R}} v_x^2\,dx\,d\tau+\varepsilon\alpha\int_0^t\int_{\mathbb{R}} u_x^2\,dx\,d\tau\\
&\quad+\frac{c_0^2\nu^2}{\varepsilon\alpha}\int_0^t\int_{\mathbb{R}} v^2\,dx\,d\tau+\delta(1-\alpha)\int_0^t\int_{\mathbb{R}} u^2\,dx\,d\tau+\frac{s^2}{\delta(1-\alpha)}\int_0^t\int_{\mathbb{R}} \bar{\theta}_x^2\,dx\,d\tau\\
&\quad+c_0\big(\|u_x\|_{L^\infty}+\|\bar{\psi}_x\|_{L^\infty}\big)\int_0^t\int_{\mathbb{R}} v^2\,dx\,d\tau+\frac{1}{2}\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}\int_0^t\int_{\mathbb{R}} v_x^2\,dx\,d\tau\\
&\quad+2c_0^2\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}^{-1}\int_0^t\int_{\mathbb{R}} \bar{\theta}^2u^2\,dx\,d\tau+c_0\delta\int_0^t\int_{\mathbb{R}} v^2\,dx\,d\tau+\frac{c_0}{\delta}\int_0^t\int_{\mathbb{R}} F^2(x,\tau)\,dx\,d\tau\\
&\quad+\big(\|u_x\|_{L^\infty}+\|\bar{\psi}_x\|_{L^\infty}\big)\int_0^t\int_{\mathbb{R}} u^2\,dx\,d\tau+\delta\int_0^t\int_{\mathbb{R}} u^2\,dx\,d\tau+\frac{1}{\delta}\int_0^t\int_{\mathbb{R}} E^2(x,\tau)\,dx\,d\tau. \qquad (3.12)
\end{aligned}
$$

Employing Lemma 2.2 and (3.9), we have from the above inequality

$$
\begin{aligned}
&\int_{\mathbb{R}}\big(u^2+c_0v^2\big)\,dx+\big\{(2-\varepsilon-\delta)(1-\alpha)-C(\delta+\delta_1)\big\}\int_0^t\int_{\mathbb{R}} u^2\,dx\,d\tau+\big\{(2-\varepsilon)\alpha-C(\delta+\delta_1)\big\}\int_0^t\int_{\mathbb{R}} u_x^2\,dx\,d\tau\\
&\quad+\left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}-C(\delta_1+\delta)\right\}\int_0^t\int_{\mathbb{R}} c_0v^2\,dx\,d\tau+\frac{1}{2}\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}\int_0^t\int_{\mathbb{R}} v_x^2\,dx\,d\tau\\
&\leqslant C\delta_0+C\delta+2c_0^2\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}^{-1}\int_0^t\int_{\mathbb{R}} \bar{\theta}^2u^2\,dx\,d\tau+\frac{c_0}{\delta}\int_0^t\int_{\mathbb{R}} F^2(x,\tau)\,dx\,d\tau\\
&\quad+\frac{1}{\delta}\int_0^t\int_{\mathbb{R}} E^2(x,\tau)\,dx\,d\tau, \qquad (3.13)
\end{aligned}
$$

where we have used the following inequality:

$$
\begin{aligned}
\frac{s^2}{\delta(1-\alpha)}\int_0^t\int_{\mathbb{R}} \bar{\theta}_x^2\,dx\,d\tau&=\frac{s^2}{\delta(1-\alpha)}\int_0^t\big\|\bar{\theta}_x(\tau)\big\|^2\,d\tau\\
&\leqslant\frac{s^2}{\delta(1-\alpha)}C\delta^2\int_0^t e^{-2(1-\alpha)\tau}\,d\tau\\
&\leqslant C\delta. \qquad (3.14)
\end{aligned}
$$

Now we estimate the last term in the right side of inequality (3.13). In fact from Lemma 2.2, we have by Cauchy–Schwartz inequality

$$
\frac{c_0}{\delta}\int_0^t\int_{\mathbb{R}} F^2(x,\tau)\,dx\,d\tau=\frac{c_0}{\delta}\int_0^t\int_{\mathbb{R}} (\nu\bar{\psi}_x+\bar{\psi}\bar{\theta}_x+\bar{\psi}_x\bar{\theta})^2\,dx\,d\tau
$$

$$\leqslant \frac{C}{\delta}\int_0^t\int_{\mathbb{R}}\left(\bar{\psi}_x^2+\bar{\theta}_x^2\right)dx\,d\tau$$
$$\leqslant C\delta. \tag{3.15}$$

We have similar estimates as follows,

$$\frac{1}{\delta}\int_0^t\int_{\mathbb{R}}E^2(x,\tau)\,dx\,d\tau\leqslant C\delta. \tag{3.16}$$

Thus, (3.13) and (3.15) give by Lemma 2.2

$$\begin{aligned}&\int_{\mathbb{R}}\left(u^2+c_0v^2\right)dx+\left\{(2-\varepsilon-\delta)(1-\alpha)-C(\delta+\delta_1)\right\}\int_0^t\int_{\mathbb{R}}u^2\,dx\,d\tau+\left\{(2-\varepsilon)\alpha-C(\delta_1+\delta)\right\}\int_0^t\int_{\mathbb{R}}u_x^2\,dx\,d\tau\\&\quad+\left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}-C(\delta+\delta_1)\right\}\int_0^t\int_{\mathbb{R}}c_0v^2\,dx\,d\tau+\frac{1}{2}\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}\int_0^t\int_{\mathbb{R}}v_x^2\,dx\,d\tau\\&\leqslant C(\delta_0+\delta)+C\int_0^t e^{-2(1-\alpha)\tau}\int_{\mathbb{R}}u^2\,dx\,d\tau,\end{aligned} \tag{3.17}$$

which implies by Lemma 2.2,

$$\int_{\mathbb{R}}u^2\,dx\leqslant C(\delta_0+\delta)+C\int_0^t e^{-2(1-\alpha)\tau}\int_{\mathbb{R}}u^2\,dx\,d\tau. \tag{3.18}$$

We easily deduce from (3.18) by Gronwall's inequality

$$\int_{\mathbb{R}}u^2\,dx\leqslant C(\delta_0+\delta)\exp\left(C\int_0^t e^{-2(1-\alpha)\tau}\,d\tau\right)\leqslant C(\delta_0+\delta). \tag{3.19}$$

Substituting (3.19) into (3.17), we have

$$\begin{aligned}&\int_{\mathbb{R}}\left(u^2+c_0v^2\right)dx+\left\{(2-\varepsilon-\delta)(1-\alpha)-C(\delta+\delta_1)\right\}\int_0^t\int_{\mathbb{R}}u^2\,dx\,d\tau+\left\{(2-\varepsilon)\alpha-C(\delta_1+\delta)\right\}\int_0^t\int_{\mathbb{R}}u_x^2\,dx\,d\tau\\&\quad+\left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}-C(\delta+\delta_1)\right\}\int_0^t\int_{\mathbb{R}}c_0v^2\,dx\,d\tau+\frac{1}{2}\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}\int_0^t\int_{\mathbb{R}}v_x^2\,dx\,d\tau\\&\leqslant C(\delta_0+\delta)+C(\delta_0+\delta)\int_0^t e^{-2(1-\alpha)\tau}\,d\tau,\end{aligned}$$

which implies by (3.10)

$$\int_{\mathbb{R}}\left(u^2+v^2\right)dx+\int_0^t\int_{\mathbb{R}}\left(u^2+v^2\right)dx\,d\tau+\int_0^t\int_{\mathbb{R}}\left(u_x^2+v_x^2\right)dx\,d\tau\leqslant C(\delta_0+\delta), \tag{3.20}$$

provided δ and δ_1 are sufficiently small. This proves Lemma 3.3. □

Lemma 3.4. *Let the assumptions in Theorem* 3.1 *hold. Then the solution* $(u(x,t), v(x,t))$ *of* (3.2), (3.3) *obtained in Lemma* 3.2 *satisfies for any* $\nu < 4\alpha(1-\alpha)$,

$$\int_{\mathbb{R}} \left(u_x^2 + v_x^2\right) dx + \int_0^t \int_{\mathbb{R}} \left(u_{xx}^2 + v_{xx}^2\right) dx\, d\tau \leqslant C(\delta + \delta_0), \tag{3.21}$$

provided δ *and* δ_1 *are sufficiently small.*

Proof. First, we multiply the first equation of (3.2) by $(-2u_{xx})$ and the second equation of (3.2) by $(-2c_0 v_{xx})$, respectively, and add the resulting equations together. After all these, we take integration over $(x,t) \in \mathbb{R} \times (0,t)$ and reach

$$\begin{aligned}
&\int_{\mathbb{R}} \left(u_x^2 + c_0 v_x^2\right) dx + 2(1-\alpha) \int_0^t \int_{\mathbb{R}} \left(u_x^2 + c_0 v_x^2\right) dx\, d\tau + 2\alpha \int_0^t \int_{\mathbb{R}} \left(u_{xx}^2 + c_0 v_{xx}^2\right) dx\, d\tau \\
&\quad = \|u_{0x}\|^2 + c_0 \|v_{0x}\|^2 + 2 \int_0^t \int_{\mathbb{R}} f_\theta u_x (v_{xx} + \bar{\theta}_{xx})\, dx\, d\tau + 2 \int_0^t \int_{\mathbb{R}} f_{\theta\theta} u_x (\bar{\theta}_x + v_x)^2\, dx\, d\tau \\
&\qquad + 2\nu c_0 \int_0^t \int_{\mathbb{R}} v_x u_{xx}\, dx\, d\tau + c_0 \int_0^t \int_{\mathbb{R}} u_x v_x^2\, dx\, d\tau + c_0 \int_0^t \int_{\mathbb{R}} \bar{\psi}_x v_x^2\, dx\, d\tau - 2c_0 \int_0^t \int_{\mathbb{R}} \bar{\theta}_x u v_{xx}\, dx\, d\tau \\
&\qquad - 2c_0 \int_0^t \int_{\mathbb{R}} u_x v v_{xx}\, dx\, d\tau - 2c_0 \int_0^t \int_{\mathbb{R}} \bar{\psi}_x v v_{xx}\, dx\, d\tau - 2c_0 \int_0^t \int_{\mathbb{R}} \bar{\theta} u_x v_{xx}\, dx\, d\tau \\
&\qquad - 2c_0 \int_0^t \int_{\mathbb{R}} F(x,\tau) v_{xx}\, dx\, d\tau - 2 \int_0^t \int_{\mathbb{R}} u_x u u_{xx}\, dx\, d\tau - 2 \int_0^t \int_{\mathbb{R}} \bar{\psi}_x u u_{xx}\, dx\, d\tau + \int_0^t \int_{\mathbb{R}} \bar{\psi}_x u_x^2\, dx\, d\tau \\
&\qquad - 2 \int_0^t \int_{\mathbb{R}} E(x,\tau) u_{xx}\, dx\, d\tau \\
&\quad \leqslant (1+c_0)\delta_0 + \varepsilon(1-\alpha) \int_0^t \int_{\mathbb{R}} u_x^2\, dx\, d\tau + \frac{s^2}{\varepsilon(1-\alpha)} \int_0^t \int_{\mathbb{R}} v_{xx}^2\, dx\, d\tau + \varepsilon\alpha \int_0^t \int_{\mathbb{R}} u_{xx}^2\, dx\, d\tau \\
&\qquad + \frac{c_0^2 \nu^2}{\varepsilon\alpha} \int_0^t \int_{\mathbb{R}} v_x^2\, dx\, d\tau + \int_0^t \int_{\mathbb{R}} u_x^2\, dx\, d\tau + s^2 \int_0^t \int_{\mathbb{R}} \bar{\theta}_{xx}^2\, dx\, d\tau + C(\delta_0 + \delta) \\
&\qquad + c_0 \left(\|u_x\|_{L^\infty} + \|\bar{\psi}_x\|_{L^\infty}\right) \int_0^t \int_{\mathbb{R}} v_x^2\, dx\, d\tau + c_0 \|\bar{\theta}_x\|_{L^\infty} \int_0^t \int_{\mathbb{R}} \left(u^2 + v_{xx}^2\right) dx\, d\tau \\
&\qquad + c_0 \|v\|_{L^\infty} \int_0^t \int_{\mathbb{R}} \left(u_x^2 + v_{xx}^2\right) dx\, d\tau + c_0 \|\bar{\psi}_x\|_{L^\infty} \int_0^t \int_{\mathbb{R}} \left(v^2 + v_{xx}^2\right) dx\, d\tau \\
&\qquad + \frac{1}{2}\left\{2c_0\alpha - \frac{s^2}{\varepsilon(1-\alpha)}\right\} \int_0^t \int_{\mathbb{R}} v_{xx}^2\, dx\, d\tau + 2c_0^2 \left\{2c_0\alpha - \frac{s^2}{\varepsilon(1-\alpha)}\right\}^{-1} \int_0^t \int_{\mathbb{R}} \bar{\theta}^2 u_x^2\, dx\, d\tau
\end{aligned}$$

$$+ c_0\delta \int_0^t \int_{\mathbb{R}} v_{xx}^2 \,dx\,d\tau + \frac{c_0}{\delta} \int_0^t \int_{\mathbb{R}} F^2(x,\tau)\,dx\,d\tau + \|u\|_{L^\infty} \int_0^t \int_{\mathbb{R}} \left(u_x^2 + u_{xx}^2\right) dx\,d\tau$$

$$+ \|\bar{\psi}_x\|_{L^\infty} \int_0^t \int_{\mathbb{R}} \left(u^2 + u_{xx}^2\right) dx\,d\tau + \|\bar{\psi}_x\|_{L^\infty} \int_0^t \int_{\mathbb{R}} u_x^2\,dx\,d\tau + \delta \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dx\,d\tau$$

$$+ \frac{1}{\delta} \int_0^t \int_{\mathbb{R}} E^2(x,\tau)\,dx\,d\tau, \tag{3.22}$$

where we have used the following inequality:

$$\begin{aligned} -2\int_0^t \int_{\mathbb{R}} f_\theta(\theta)\theta_x u_{xx}\,dx\,d\tau &= 2\int_0^t \int_{\mathbb{R}} \left(f_{\theta\theta}(\theta)(\theta_x)^2 + f_\theta(\theta)\theta_{xx}\right)u_x\,dx\,d\tau \\ &= 2\int_0^t \int_{\mathbb{R}} \left(f_{\theta\theta}(\theta)(v_x+\bar{\theta}_x)^2 + f_\theta(\theta)(v_{xx}+\bar{\theta}_{xx})\right)u_x\,dx\,d\tau \\ &\leqslant C(\delta_0+\delta) + \int_0^t \int_{\mathbb{R}} f_\theta(\theta)(v_{xx}+\bar{\theta}_{xx})u_x\,dx\,d\tau. \end{aligned} \tag{3.23}$$

By Lemmas 2.2 and 3.3, we have from (3.15) and the above inequalities

$$\begin{aligned} &\int_{\mathbb{R}} \left(u_x^2 + c_0 v_x^2\right) dx + \left\{(2-\varepsilon)\alpha - C\delta\right\} \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dx\,d\tau + \frac{1}{2}\left\{2c_0\alpha - \frac{1}{\varepsilon(1-\alpha)} - C(\delta+\delta_1)\right\} \int_0^t \int_{\mathbb{R}} v_{xx}^2\,dx\,d\tau \\ &\quad \leqslant C(\delta_0+\delta), \end{aligned} \tag{3.24}$$

which implies by (3.10)

$$\int_{\mathbb{R}} \left(u_x^2 + v_x^2\right) dx + \int_0^t \int_{\mathbb{R}} \left(u_{xx}^2 + v_{xx}^2\right) dx\,d\tau \leqslant C(\delta+\delta_0), \tag{3.25}$$

provided δ and δ_1 are sufficiently small. This proves Lemma 3.4. □

Lemma 3.5. *Suppose that $(u(x,t), v(x,t))$ is a solution to* (3.2), (3.3) *obtained in Lemma* 3.2 *under the assumptions in Theorem* 3.1*, then for any $\nu < 4\alpha(1-\alpha)$, we have*

$$\int_{\mathbb{R}} \left(u_{xx}^2 + v_{xx}^2\right) dx + \int_0^t \int_{\mathbb{R}} \left(u_{xxx}^2 + v_{xxx}^2\right) dx\,d\tau \leqslant C(\delta+\delta_0), \tag{3.26}$$

provided δ and δ_1 are sufficiently small.

Proof. Differentiating (3.2) twice with respect to x, multiplying the results by $2u_{xx}$ and $2c_0 v_{xx}$, respectively, integrating the resulting equations with respect to (x,t) over $\mathbb{R} \times (0,t)$, we get

$$\begin{aligned} &\int_{\mathbb{R}} \left(u_{xx}^2 + c_0 v_{xx}^2\right) dx + 2(1-\alpha) \int_0^t \int_{\mathbb{R}} \left(u_{xx}^2 + c_0 v_{xx}^2\right) dx\,d\tau + 2\alpha \int_0^t \int_{\mathbb{R}} \left(u_{xxx}^2 + c_0 v_{xxx}^2\right) dx\,d\tau \\ &\quad = \|u_{0xx}\|^2 + c_0\|v_{0xx}\|^2 + 2\int_0^t \int_{\mathbb{R}} f_\theta u_{xx} v_{xxx}\,dx\,d\tau + 2\nu c_0 \int_0^t \int_{\mathbb{R}} v_{xx} u_{xxx}\,dx\,d\tau \end{aligned}$$

$$
\begin{aligned}
&+2\int_0^t\int_{\mathbb{R}} f_\theta u_{xx}\bar{\theta}_{xxx}\,dx\,d\tau+2c_0\int_0^t\int_{\mathbb{R}}(uv)_{xxx}v_{xx}\,dx\,d\tau+2c_0\int_0^t\int_{\mathbb{R}}(\bar{\psi}v)_{xxx}v_{xx}\,dx\,d\tau\\
&+2c_0\int_0^t\int_{\mathbb{R}}(\bar{\theta}u)_{xxx}v_{xx}\,dx\,d\tau+2c_0\int_0^t\int_{\mathbb{R}}F_{xx}(x,\tau)v_{xx}\,dx\,d\tau\\
&+\int_0^t\int_{\mathbb{R}}(u^2)_{xxx}u_{xx}\,dx\,d\tau+2\int_0^t\int_{\mathbb{R}}(\bar{\psi}u)_{xxx}u_{xx}\,dx\,d\tau+2\int_0^t\int_{\mathbb{R}}E_{xx}(x,\tau)u_{xx}\,dx\,d\tau\\
&\leqslant(1+c_0)\delta_0+\varepsilon(1-\alpha)\int_0^t\int_{\mathbb{R}}u_{xx}^2\,dx\,d\tau+\frac{s^2}{\varepsilon(1-\alpha)}\int_0^t\int_{\mathbb{R}}v_{xxx}^2\,dx\,d\tau+\varepsilon\alpha\int_0^t\int_{\mathbb{R}}u_{xxx}^2\,dx\,d\tau\\
&+\frac{c_0^2\nu^2}{\varepsilon\alpha}\int_0^t\int_{\mathbb{R}}v_{xx}^2\,dx\,d\tau+\int_0^t\int_{\mathbb{R}}u_{xx}^2\,dx\,d\tau+s^2\int_0^t\int_{\mathbb{R}}\bar{\theta}_{xxx}^2\,dx\,d\tau-2c_0\int_0^t\int_{\mathbb{R}}(uv)_{xx}v_{xxx}\,dx\,d\tau\\
&-2c_0\int_0^t\int_{\mathbb{R}}(\bar{\psi}v)_{xx}v_{xxx}\,dx\,d\tau-2c_0\int_0^t\int_{\mathbb{R}}(\bar{\theta}u)_{xx}v_{xxx}\,dx\,d\tau-2c_0\int_0^t\int_{\mathbb{R}}F_x(x,\tau)v_{xxx}\,dx\,d\tau\\
&-\int_0^t\int_{\mathbb{R}}(u^2)_{xx}v_{xxx}\,dx\,d\tau-2\int_0^t\int_{\mathbb{R}}(\bar{\psi}u)_{xx}u_{xxx}\,dx\,d\tau-2\int_0^t\int_{\mathbb{R}}E_x(x,\tau)v_{xxx}\,dx\,d\tau,
\end{aligned}
\tag{3.27}
$$

where we have used the following inequality:

$$
\begin{aligned}
2\int_0^t\int_{\mathbb{R}} f_{xxx}(\theta)u_{xx}&=2\int_0^t\int_{\mathbb{R}}\big(f_{\theta\theta\theta}(\theta_x)^3+3f_\theta\theta_x\theta_{xx}+f_\theta\theta_{xxx}\big)u_{xx}\\
&=2\int_0^t\int_{\mathbb{R}}\big(f_{\theta\theta\theta}(v_x+\bar{\theta}_x)^3+3f_\theta(v_x+\bar{\theta}_x)(v_{xx}+\bar{\theta}_{xx})_{xx}+f_\theta(v_{xxx}+\bar{\theta}_{xxx})\big)u_{xx}\\
&\leqslant C(\delta_0+\delta)+f_\theta(v_{xxx}+\bar{\theta}_{xxx})u_{xx}.
\end{aligned}
\tag{3.28}
$$

Shuffling the terms, we get by Lemmas 2.2 and 3.4

$$
\begin{aligned}
&\int_{\mathbb{R}}\big(u_{xx}^2+c_0v_{xx}^2\big)\,dx+(2-\varepsilon)\alpha\int_0^t\int_{\mathbb{R}}u_{xxx}^2\,dx\,d\tau+\left\{2c_0\alpha-\frac{1}{\varepsilon(1-\alpha)}\right\}\int_0^t\int_{\mathbb{R}}v_{xxx}^2\,dx\,d\tau\\
&\leqslant C(\delta_0+\delta)-2c_0\int_0^t\int_{\mathbb{R}}(uv)_{xx}v_{xxx}\,dx\,d\tau-2c_0\int_0^t\int_{\mathbb{R}}(\bar{\psi}v)_{xx}v_{xxx}\,dx\,d\tau\\
&-2c_0\int_0^t\int_{\mathbb{R}}(\bar{\theta}u)_{xx}v_{xxx}\,dx\,d\tau-2c_0\int_0^t\int_{\mathbb{R}}F_x(x,\tau)v_{xxx}\,dx\,d\tau\\
&-\int_0^t\int_{\mathbb{R}}\big(u^2\big)_{xx}v_{xxx}\,dx\,d\tau-2\int_0^t\int_{\mathbb{R}}(\bar{\psi}u)_{xx}u_{xxx}\,dx\,d\tau-2\int_0^t\int_{\mathbb{R}}E_x(x,\tau)v_{xxx}\,dx\,d\tau.
\end{aligned}
\tag{3.29}
$$

Next we are devoted to estimate the terms in the right side of (3.29) as follows.

First, we obtain from (3.9), Lemmas 3.3 and 3.4 by Cauchy–Schwartz inequality

$$
\begin{aligned}
-2c_0\int_0^t\int_{\mathbb{R}}(uv)_{xx}v_{xxx}\,dx\,d\tau &\leqslant \delta_1^2\int_0^t\int_{\mathbb{R}} v_{xxx}^2\,dx\,d\tau+\frac{C}{\delta_1^2}\int_0^t\int_{\mathbb{R}}\big(v^2u_{xx}^2+u_x^2v_x^2+u^2v_{xx}^2\big)\,dx\,d\tau\\
&\leqslant \delta_1^2\int_0^t\int_{\mathbb{R}} v_{xxx}^2\,dx\,d\tau+C(\delta+\delta_0).
\end{aligned}\tag{3.30}
$$

We also have

$$
\int_0^t\int_{\mathbb{R}}(u^2)_{xx}u_{xxx}\,dx\,d\tau\leqslant\delta_1^2\int_0^t\int_{\mathbb{R}}u_{xxx}^2\,dx\,d\tau+C(\delta+\delta_0).\tag{3.31}
$$

Similarly, we have from Lemmas 2.2, 3.3 and 3.4,

$$
\begin{aligned}
-2c_0\int_0^t\int_{\mathbb{R}}(\bar\psi v)_{xx}v_{xxx}\,dx\,d\tau &= -2c_0\int_0^t\int_{\mathbb{R}}(\bar\psi v_{xx}+2\bar\psi_x v_x+\bar\psi_{xx}v)v_{xxx}\,dx\,d\tau\\
&= c_0\int_0^t\int_{\mathbb{R}}\bar\psi_x v_{xx}^2\,dx\,d\tau-2c_0\int_0^t\int_{\mathbb{R}}(2\bar\psi_x v_x+\bar\psi_{xx}v)v_{xxx}\,dx\,d\tau\\
&\leqslant c_0\int_0^t\int_{\mathbb{R}}\bar\psi_x v_{xx}^2\,dx\,d\tau+\delta\int_0^t\int_{\mathbb{R}}v_{xxx}^2\,dx\,d\tau+\frac{C}{\delta}\int_0^t\int_{\mathbb{R}}\big(\bar\psi_x^2v_x^2+\bar\psi_{xx}^2v^2\big)\,dx\,d\tau\\
&\leqslant \delta\int_0^t\int_{\mathbb{R}}v_{xxx}^2\,dx\,d\tau+C(\delta+\delta_0)
\end{aligned}\tag{3.32}
$$

and

$$
-2c_0\int_0^t\int_{\mathbb{R}}(\bar\theta u)_{xx}v_{xxx}\,dx\,d\tau\leqslant\delta\int_0^t\int_{\mathbb{R}}v_{xxx}^2\,dx\,d\tau+C(\delta+\delta_0).\tag{3.33}
$$

Also

$$
-2c_0\int_0^t\int_{\mathbb{R}}(\bar\psi u)_{xx}u_{xxx}\,dx\,d\tau\leqslant\delta\int_0^t\int_{\mathbb{R}}u_{xxx}^2\,dx\,d\tau+C(\delta+\delta_0).\tag{3.34}
$$

In addition, applying Lemmas 2.2, 3.3 and 3.4, we derive by Cauchy–Schwartz inequality

$$
\begin{aligned}
&-2c_0\int_0^t\int_{\mathbb{R}}F_x(x,\tau)v_{xxx}\,dx\,d\tau\\
&\quad=-2c_0\int_0^t\int_{\mathbb{R}}(\nu\bar\psi_x+\bar\psi\bar\theta_x+\bar\psi_x\bar\theta)_x v_{xxx}\,dx\,d\tau\\
&\quad\leqslant\delta\int_0^t\int_{\mathbb{R}}v_{xxx}^2\,dx\,d\tau+\frac{c_0^2}{\delta}\int_0^t\int_{\mathbb{R}}\big(\nu\bar\psi_{xx}+(\bar\psi\bar\theta_x)_x+(\bar\psi_x\bar\theta)_x\big)^2\,dx\,d\tau
\end{aligned}
$$

$$\leqslant \delta \int_0^t \int_{\mathbb{R}} v_{xxx}^2 \,dx\,d\tau + \frac{C}{\delta} \int_0^t \int_{\mathbb{R}} \bar{\psi}_{xx}^2 \,dx\,d\tau + \frac{C}{\delta} \int_0^t \int_{\mathbb{R}} \bar{\psi}_x^2 \bar{\theta}_x^2 \,dx\,d\tau + \frac{C}{\delta} \int_0^t \int_{\mathbb{R}} \bar{\psi}^2 \bar{\theta}_{xx}^2 \,dx\,d\tau$$

$$\leqslant \delta \int_0^t \int_{\mathbb{R}} v_{xxx}^2 \,dx\,d\tau + C(\delta + \delta_0) \tag{3.35}$$

and

$$-2 \int_0^t \int_{\mathbb{R}} E_x(x,\tau) u_{xxx} \,dx\,d\tau \leqslant \delta \int_0^t \int_{\mathbb{R}} u_{xxx}^2 \,dx\,d\tau + C(\delta + \delta_0). \tag{3.36}$$

Substituting (3.26)–(3.29) into (3.25), we get

$$\int_{\mathbb{R}} \left(u_{xx}^2 + c_0 v_{xx}^2\right) dx + \left\{(2-\varepsilon)\alpha - \left(\delta_1^2 + 3\delta\right)\right\} \int_0^t \int_{\mathbb{R}} u_{xxx}^2 \,dx\,d\tau$$
$$+ \left\{2c_0\alpha - \frac{s^2}{\varepsilon(1-\alpha)} - \left(\delta_1^2 + 3\delta\right)\right\} \int_0^t \int_{\mathbb{R}} v_{xxx}^2 \,dx\,d\tau \leqslant C(\delta + \delta_0), \tag{3.37}$$

which proves (3.26) from (3.10) provided δ and δ_1 are sufficiently small. Thus the combination of Lemmas 3.3–3.5 implies (3.5). □

Finally, we have to show that the a priori assumption (3.8) can be closed. Since, under this a priori assumption (3.8), we deduced that (3.5) holds provided δ and δ_1 are sufficiently small. Therefore the assumption (3.8) is always true provided δ and δ_0 are sufficiently small.

Now we turn to show that (3.6) is true. To do this, we introduce the following lemma.

Lemma 3.6. *If $g(t) \geqslant 0$, $g(t) \in L^1(0,\infty)$ and $g'(t) \in L^1(0,\infty)$, then $g(t) \to 0$ as $t \to \infty$.*

The proof of Lemma 3.6 can be found in [13,18] and the details are omitted.

Taking $g(t) = \|u_x(t)\|^2$ in Lemma 3.6, we can conclude from (3.5) that $g(t) \in L^1(0,\infty)$. Denote L^2-inner product by $\langle\cdot,\cdot\rangle$. By using the definition of L^2-inner product and integrating by parts, we have $g'(t) = 2\langle u_x, u_{xt}\rangle = -2\langle u_t, u_{xx}\rangle$. It is easy to verify from the estimates (3.5) and Cauchy–Schwartz inequality that

$$-\langle u_t, u_{xx}\rangle = \langle -u_t, u_{xx}\rangle \in L^1(0,\infty).$$

Hence, $g'(t) \in L^1(0,\infty)$ which implies by Lemma 3.6

$$\left\|u_x(t)\right\| \to 0 \quad \text{as } t \to \infty. \tag{3.38}$$

Applying Sobolev inequality, we have from (3.38) and (3.5)

$$\sup\left|u(x,t)\right| \leqslant \left\|u(t)\right\|^{\frac{1}{2}} \left\|u_x(t)\right\|^{\frac{1}{2}} \to 0 \quad \text{as } t \to \infty. \tag{3.39}$$

The same process applied to $v(x,t)$ deduces that

$$\sup\left|v(x,t)\right| \leqslant \left\|v(t)\right\|^{\frac{1}{2}} \left\|v_x(t)\right\|^{\frac{1}{2}} \to 0 \quad \text{as } t \to \infty. \tag{3.40}$$

Similarly, taking $g(t) = \|u_{xx}(t)\|^2$ and $g(t) = \|v_{xx}(t)\|^2$, we have from (3.5) and Lemma 3.6

$$\left\|u_{xx}(t)\right\| \to 0 \quad \text{and} \quad \left\|v_{xx}(t)\right\| \to 0, \quad \text{as } t \to \infty. \tag{3.41}$$

So Sobolev inequality and (3.41) give

$$\sup\left|u_x(x,t)\right| \leqslant \left\|u_x(t)\right\|^{\frac{1}{2}} \left\|u_{xx}(t)\right\|^{\frac{1}{2}} \to 0 \quad \text{as } t \to \infty, \tag{3.42}$$

and

$$\sup\left|v_x(x,t)\right| \leqslant \left\|v_x(t)\right\|^{\frac{1}{2}}\left\|v_{xx}(t)\right\|^{\frac{1}{2}} \to 0 \quad \text{as } t \to \infty. \tag{3.43}$$

Thus, (3.6) is proved by (3.39), (3.40), (3.42) and (3.43).

The proof of Theorem 3.1 is completed.

4. Decay rates

In this section, we will study the decay rates of solutions to the Cauchy problem (3.2), (3.3) under a priori assumption

$$\sum_{k=0}^{2}\left\|\partial_x^k u(t)\right\|^2 + \sum_{k=0}^{2}\left\|\partial_x^k v(t)\right\|^2 \leqslant e^{-lt}, \quad 0 < t < T, \tag{4.1}$$

with

$$l = \min\left\{(2-\varepsilon)(1-\alpha), 2(1-\alpha) - \frac{c_0\nu^2}{\varepsilon\alpha}\right\}, \tag{4.2}$$

where ε and c_0 are defined by (3.10).

By Sobolev inequality, we have from (4.1)

$$\left\|(u, u_x, v, v_x)\right\|_{L^\infty} \leqslant e^{-\frac{l}{2}t}, \tag{4.3}$$

which will be used later.

Moreover, we list the following Gronwall's inequality which is used in the following text.

Lemma 4.1 *(Gronwall's inequality). Let $\eta(\cdot)$ be a nonnegative continuous function on $[0,\infty)$, which satisfies the differential inequality*

$$\eta'(t) + \lambda\eta(t) \leqslant \omega(t),$$

where λ is a positive constant and $\omega(t)$ is a nonnegative continuous function on $[0,\infty)$. Then

$$\eta(t) \leqslant \left(\eta(0) + \int_0^t e^{\lambda\tau}\omega(\tau)\,d\tau\right)e^{-\lambda t}.$$

Now we can state the main results of decay rates of solutions.

Theorem 4.2. *Suppose that $(u(x,t), v(x,t))$ is a solution to problem (3.2), (3.3) under the assumptions imposed in Theorem 3.1, then when $\nu < 4\alpha(1-\alpha)$, we have for any $t \in [0,T]$,*

$$\left\|\partial_x^k u(t)\right\|^2 + \left\|\partial_x^k v(t)\right\|^2 \leqslant C(\delta+\delta_0)^{\frac{1}{4}}e^{-lt}, \quad k = 0, 1, 2, \tag{4.4}$$

provided δ and δ_0 are sufficiently small, where l is defined by (4.2).

Proof. The proof is divided into three steps.

First, $(3.2)_1 \times 2u + (3.2)_2 \times 2c_0 v$ and integrating the resulting identities over $x \in \mathbb{R}$, we reach by Cauchy–Schwartz inequality from Lemma 2.2

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left(u^2 + c_0v^2\right)dx + 2(1-\alpha)\int_{\mathbb{R}}\left(u^2 + c_0v^2\right)dx + 2\alpha\int_{\mathbb{R}}\left(u_x^2 + c_0v_x^2\right)dx \\
&\quad = +2\int_{\mathbb{R}} f_\theta u v_x\,dx + 2\nu c_0\int_{\mathbb{R}} v u_x\,dx + 2\int_{\mathbb{R}} f_\theta u\bar{\theta}_x\,dx - 2c_0\int_{\mathbb{R}} u v v_x\,dx - 2c_0\int_{\mathbb{R}} \bar{\theta} u v_x\,dx
\end{aligned}$$

$$
\begin{aligned}
&-2c_0\int_{\mathbb{R}}\bar{\psi}vv_x\,dx+2c_0\int_{\mathbb{R}}vF(x,t)\,dx-2\int_{\mathbb{R}}u_xu^2\,dx-2\int_{\mathbb{R}}\bar{\psi}u_xu\,dx+2\int_{\mathbb{R}}uE(x,t)\,dx\\
&\leqslant\varepsilon(1-\alpha)\int_{\mathbb{R}}u^2\,dx+\frac{s^2}{\varepsilon(1-\alpha)}\int_{\mathbb{R}}v_x^2\,dx+\varepsilon\alpha\int_{\mathbb{R}}u_x^2\,dx+\frac{c_0^2\nu^2}{\varepsilon\alpha}\int_{\mathbb{R}}v^2\,dx-2\int_{\mathbb{R}}f_\theta u\bar{\theta}_x\,dx\\
&\quad+c_0\int_{\mathbb{R}}u_xv^2\,dx-2c_0\int_{\mathbb{R}}\bar{\theta}uv_x\,dx+c_0\int_{\mathbb{R}}\bar{\psi}_xv^2\,dx+2c_0\int_{\mathbb{R}}vF(x,t)\,dx\\
&\quad+\int_{\mathbb{R}}u_xu^2\,dx+\int_{\mathbb{R}}\bar{\psi}_xu^2\,dx+2\int_{\mathbb{R}}uE(x,t)\,dx. \qquad (4.5)
\end{aligned}
$$

After shuffling the terms, we have by Lemma 2.2

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left(u^2+c_0v^2\right)dx+(2-\varepsilon)(1-\alpha)\int_{\mathbb{R}}u^2\,dx+\left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}\right\}\int_{\mathbb{R}}c_0v^2\,dx\\
&\quad+(2-\varepsilon)\alpha\int_{\mathbb{R}}u_x^2\,dx+\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}\int_{\mathbb{R}}v_x^2\,dx\\
&\leqslant C\delta e^{-\{l+(1-\alpha)\}t}+2\int_{\mathbb{R}}f_\theta u\bar{\theta}_x\,dx+c_0\int_{\mathbb{R}}u_xv^2\,dx-2c_0\int_{\mathbb{R}}\bar{\theta}uv_x\,dx+2c_0\int_{\mathbb{R}}vF(x,t)\,dx\\
&\quad+\int_{\mathbb{R}}u_xu^2\,dx+\int_{\mathbb{R}}\bar{\psi}_xu^2\,dx+2\int_{\mathbb{R}}uE(x,t)\,dx. \qquad (4.6)
\end{aligned}
$$

Next, we will estimate the terms in the right side of (4.6).

In fact, we have from the assumption (4.1) and Lemma 2.2 by employing Cauchy–Schwartz inequality

$$
\begin{aligned}
2\int_{\mathbb{R}}f_\theta u\bar{\theta}_x\,dx&\leqslant\delta e^{\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}}u^2\,dx+\frac{s^2}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}}\bar{\theta}_x^2\,dx\\
&\leqslant\delta e^{\{\frac{l}{2}-(1-\alpha)\}t}e^{-lt}+C\delta e^{-\{\frac{l}{2}-(1-\alpha)\}t}e^{-2(1-\alpha)t}\\
&\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}. \qquad (4.7)
\end{aligned}
$$

Moreover, we get from (3.5) by Sobolev inequality

$$\left\|(u,u_x,v,v_x)\right\|_{L^\infty}\leqslant C(\delta+\delta_0)^{\frac{1}{2}}. \qquad (4.8)$$

Thus we derive by Cauchy–Schwartz inequality from (4.1), (4.3) and (4.8)

$$
\begin{aligned}
c_0\int_{\mathbb{R}}u_xv^2\,dx&\leqslant c_0\left(\int_{\mathbb{R}}u_x^2|v|\,dx+\int_{\mathbb{R}}|v|^3\,dx\right)\\
&\leqslant c_0\|v\|_{L^\infty}\int_{\mathbb{R}}\left(u_x^2+v^2\right)dx\\
&=c_0\|v\|_{L^\infty}^{\frac{1}{2}}\|v\|_{L^\infty}^{\frac{1}{2}}\int_{\mathbb{R}}\left(u_x^2+v^2\right)dx\\
&\leqslant C(\delta+\delta_0)^{\frac{1}{4}}e^{-\frac{l}{4}t}e^{-lt}\\
&=C(\delta+\delta_0)^{\frac{1}{4}}e^{-\frac{5l}{4}t} \qquad (4.9)
\end{aligned}
$$

and

$$\int_{\mathbb{R}} u_x u^2\,dx \leqslant C(\delta+\delta_0)^{\frac{1}{4}} e^{-\frac{5l}{4}t}. \tag{4.10}$$

Similarly, we get the following inequality:

$$\begin{aligned}
-2c_0\int_{\mathbb{R}} \bar\theta u v_x\,dx &\leqslant \frac{1}{2}\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}\int_{\mathbb{R}} v_x^2\,dx\,d\tau + C\int_{\mathbb{R}} \bar\theta^2 u^2\,dx \\
&\leqslant \frac{1}{2}\left\{2c_0\alpha-\frac{s^2}{\varepsilon(1-\alpha)}\right\}\int_{\mathbb{R}} v_x^2\,dx\,d\tau + Ce^{-2(1-\alpha)t}\int_{\mathbb{R}} u^2\,dx.
\end{aligned} \tag{4.11}$$

In addition, we deduce from Lemma 2.2 and Cauchy–Schwartz inequality

$$\begin{aligned}
2c_0\int_{\mathbb{R}} vF(x,t)\,dx &\leqslant \delta e^{\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}} v^2\,dx + \frac{c_0^2}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}} F^2(x,t)\,dx \\
&\leqslant \delta e^{\{\frac{l}{2}-(1-\alpha)\}t}e^{-lt} + \frac{C}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}}\left(\bar\psi_x^2+\bar\theta_x^2\right)dx \\
&\leqslant \delta e^{\{\frac{l}{2}-(1-\alpha)\}t}e^{-lt} + \frac{C}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\delta^2 e^{-2(1-\alpha)t} \\
&\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}.
\end{aligned} \tag{4.12}$$

Similarly, we get

$$2\int_{\mathbb{R}} uE(x,t)\,dx \leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}. \tag{4.13}$$

Thus, we get from (4.6), (4.7), (4.9), (4.11), (4.12) and (4.1)

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left(u^2+c_0v^2\right)dx + (2-\varepsilon)(1-\alpha)\int_{\mathbb{R}} u^2\,dx + \left\{2(1-\alpha)-\frac{c_0v^2}{\varepsilon\alpha}\right\}\int_{\mathbb{R}} c_0v^2\,dx \\
&\quad\leqslant C\delta e^{-\{l+(1-\alpha)\}t} + C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + C(\delta+\delta_0)^{\frac{1}{4}}e^{-\frac{5l}{4}t} + Ce^{-2(1-\alpha)t}\int_{\mathbb{R}} u^2\,dx \\
&\quad\leqslant C(\delta+\delta_0)^{\frac{1}{4}}e^{-\frac{5l}{4}t} + C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + Ce^{-2(1-\alpha)t}\int_{\mathbb{R}} u^2\,dx.
\end{aligned} \tag{4.14}$$

Recalling the definition (4.2) of l, we have from (4.14)

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2+c_0v^2\right)dx + l\int_{\mathbb{R}}\left(u^2+c_0v^2\right)dx \leqslant C(\delta+\delta_0)^{\frac{1}{4}}e^{-\frac{5l}{4}t} + C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + Ce^{-2(1-\alpha)t}\int_{\mathbb{R}} v^2\,dx. \tag{4.15}$$

Noticing that $\frac{l}{2} < 1-\alpha$, we obtain from Lemmas 4.1 and 3.3

$$\begin{aligned}
\int_{\mathbb{R}}(u^2+c_0v^2)\,dx &\leqslant \left\{C\delta_0 + C(\delta+\delta_0)^{\frac{1}{4}}\int_0^t e^{-\frac{5l}{4}\tau}e^{l\tau}\,d\tau\right\}e^{-lt} \\
&\quad + \left\{C\delta\int_0^t e^{-\{\frac{l}{2}+(1-\alpha)\}\tau}e^{l\tau}\,d\tau + C\int_0^t e^{l\tau}e^{-2(1-\alpha)\tau}\int_{\mathbb{R}} v^2\,dx\,d\tau\right\}e^{-lt} \\
&\leqslant C(\delta+\delta_0)^{\frac{1}{4}}e^{-lt},
\end{aligned} \tag{4.16}$$

which implies (4.4) for $k=0$.

Similarly, for $k=1$, if we multiply $(3.2)_1$ by $(-2u_{xx})$ and $(3.2)_2$ by $(-2c_0 v_{xx})$, add and take integration of the resulting identities over $x\in\mathbb{R}$, then we get by conducting the same procedure to the proofs of inequality (4.14)

$$\int_{\mathbb{R}} \left(u_x^2 + c_0 v_x^2\right) dx \leqslant C(\delta+\delta_0)^{\frac{1}{4}} e^{-lt}, \tag{4.17}$$

which proves (4.4) for $k=1$.

Finally, we shall show (4.4) is true for $k=2$. In fact, differentiating (3.2) twice with respect to x, multiplying the results by $2u_{xx}$ and $2c_0 v_{xx}$, respectively, integrating the resulting equation over $x\in\mathbb{R}$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \left(u_{xx}^2 + c_0 v_{xx}^2\right) dx + 2(1-\alpha)\int_{\mathbb{R}} \left(u_{xx}^2 + c_0 v_{xx}^2\right) dx + 2\alpha \int_{\mathbb{R}} \left(u_{xxx}^2 + c_0 v_{xxx}^2\right) dx \\
&\quad = -2\int_{\mathbb{R}} u_{xx} v_{xxx}\, dx + 2\nu c_0 \int_{\mathbb{R}} f_\theta v_{xx} u_{xxx}\, dx + 2\int_{\mathbb{R}} f_\theta u_{xx} \bar\theta_{xxx}\, dx + 2c_0 \int_{\mathbb{R}} (uv)_{xxx} v_{xx}\, dx \\
&\qquad + 2c_0 \int_{\mathbb{R}} (\bar\psi v)_{xxx} v_{xx}\, dx + 2c_0 \int_{\mathbb{R}} (\bar\theta u)_{xxx} v_{xx}\, dx \\
&\qquad + 2c_0 \int_{\mathbb{R}} F_{xx}(x,t) v_{xx}\, dx + 2\int_{\mathbb{R}} E_{xx}(x,t) u_{xx}\, dx \\
&\qquad + \int_{\mathbb{R}} (u^2)_{xxx} v_{xx}\, dx + 2\int_{\mathbb{R}} (\bar\psi u)_{xxx} u_{xx}\, dx \\
&\quad \leqslant \varepsilon(1-\alpha)\int_{\mathbb{R}} u_{xx}^2\, dx + \frac{s^2}{\varepsilon(1-\alpha)}\int_{\mathbb{R}} v_{xxx}^2\, dx + \varepsilon\alpha \int_{\mathbb{R}} u_{xxx}^2\, dx + \frac{c_0^2 \nu^2}{\varepsilon\alpha}\int_{\mathbb{R}} v_{xx}^2\, dx \\
&\qquad - 2\int_{\mathbb{R}} f_\theta u_{xx} \bar\theta_{xxx}\, dx - 2c_0 \int_{\mathbb{R}} (uv)_{xx} v_{xxx}\, dx - 2c_0 \int_{\mathbb{R}} (\bar\psi v)_{xx} v_{xxx}\, dx \\
&\qquad - 2c_0 \int_{\mathbb{R}} (\bar\theta u)_{xx} v_{xxx}\, dx + 2c_0 \int_{\mathbb{R}} F_{xx}(x,t) v_{xx}\, dx + 2\int_{\mathbb{R}} E_{xx}(x,t) u_{xx}\, dx \\
&\qquad - \int_{\mathbb{R}} \left(u^2\right)_{xx} u_{xxx}\, dx - 2\int_{\mathbb{R}} (\bar\psi u)_{xx} u_{xxx}\, dx,
\end{aligned} \tag{4.18}$$

where we have used the following inequality:

$$\begin{aligned}
2\int_{\mathbb{R}} f_{xxx}(\theta) u_{xx} &= 2\int_{\mathbb{R}} \left(f_{\theta\theta\theta}(\theta)(\theta_x)^3 + 3 f_\theta(\theta)\theta_x\theta_{xx} + f_\theta(\theta)\theta_{xxx}\right) u_{xx} \\
&= 2\int_{\mathbb{R}} \left(f_{\theta\theta\theta}(\theta)(v_x+\bar\theta_x)^3 + 3 f_\theta(\theta)(v_x+\bar\theta_x)(v_{xx}+\bar\theta_{xx})_{xx} + f_\theta(\theta)(v_{xxx}+\bar\theta_{xxx})\right) u_{xx} \\
&\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + C(\delta+\delta_0)e^{-2(1-\alpha)t} + Ce^{-lt}\int_{\mathbb{R}} u_{xx}^2\, dx + 2\int_{\mathbb{R}} f_\theta(\theta)(v_{xxx}+\bar\theta_{xxx}) u_{xx}.
\end{aligned} \tag{4.19}$$

(4.18) implies

$$\frac{d}{dt}\int_{\mathbb{R}} \left(u_{xx}^2 + c_0 v_{xx}^2\right) dx + (2-\varepsilon)(1-\alpha)\int_{\mathbb{R}} u_{xx}^2\, dx + \left\{2(1-\alpha) - \frac{c_0\nu^2}{\varepsilon\alpha}\right\}\int_{\mathbb{R}} c_0 v_{xx}^2\, dx$$

$$
\begin{aligned}
&+ (2-\varepsilon)\alpha \int_{\mathbb{R}} u_{xxx}^2\,dx + \left\{2c_0\alpha - \frac{s^2}{\varepsilon(1-\alpha)}\right\}\int_{\mathbb{R}} v_{xxx}^2\,dx \\
\leqslant{}& +2\int_{\mathbb{R}} f_\theta u_{xx}\bar{\theta}_{xxx}\,dx - 2c_0\int_{\mathbb{R}} (uv)_{xx}v_{xxx}\,dx - 2c_0\int_{\mathbb{R}} (\bar{\psi}v)_{xx}v_{xxx}\,dx \\
&- 2c_0\int_{\mathbb{R}} (\bar{\theta}u)_{xx}v_{xxx}\,dx + 2c_0\int_{\mathbb{R}} F_{xx}(x,t)v_{xx}\,dx + 2\int_{\mathbb{R}} E_{xx}(x,t)u_{xx}\,dx \\
&- \int_{\mathbb{R}} \left(u^2\right)_{xx}u_{xxx}\,dx - 2\int_{\mathbb{R}} (\bar{\psi}u)_{xx}u_{xxx}\,dx.
\end{aligned}
\tag{4.20}
$$

Next, we are going to give the estimates of the terms in the right side of (4.20).

Indeed, using Cauchy–Schwartz inequality, we have from (4.1) and Lemma 2.2

$$
\begin{aligned}
2\int_{\mathbb{R}} f_\theta u_{xx}\bar{\theta}_{xxx}\,dx &\leqslant \delta e^{\{\frac{1}{2}-(1-\alpha)\}t}\int_{\mathbb{R}} u_{xx}^2\,dx + \frac{s^2}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}} \bar{\theta}_{xxx}^2\,dx \\
&\leqslant \delta e^{\{\frac{1}{2}-(1-\alpha)\}t}e^{-lt} + \frac{s^2}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}C\delta^2 e^{-2(1-\alpha)t} \\
&\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}.
\end{aligned}
\tag{4.21}
$$

For the simplicity of notation, we define $\lambda = 2c_0\alpha - \frac{1}{\varepsilon(1-\alpha)}$, $\mu = \alpha(2-\varepsilon)$. Then we derive from (4.3) by Cauchy–Schwartz inequality

$$
\begin{aligned}
&-2c_0\int_{\mathbb{R}} (uv)_{xx}v_{xxx}\,dx \\
&\leqslant \frac{1}{16}\lambda\int_{\mathbb{R}} v_{xxx}^2\,dx + C\int_{\mathbb{R}} \left(u_x^2v_x^2 + u^2v_{xx}^2 + v^2u_{xx}^2\right)dx \\
&\leqslant \frac{1}{16}\lambda\int_{\mathbb{R}} v_{xxx}^2\,dx + C\|u_x\|_{L^\infty}^2\int_{\mathbb{R}} v_x^2\,dx + C\|u\|_{L^\infty}^2\int_{\mathbb{R}} v_{xx}^2\,dx + C\|v\|_{L^\infty}^2\int_{\mathbb{R}} u_{xx}^2\,dx \\
&\leqslant \frac{1}{16}\lambda\int_{\mathbb{R}} v_{xxx}^2\,dx + Ce^{-lt}\int_{\mathbb{R}} \left(v_x^2 + v_{xx}^2 + u_{xx}^2\right)dx.
\end{aligned}
\tag{4.22}
$$

We also have estimate

$$
\int_{\mathbb{R}} \left(u^2\right)_{xx}u_{xxx}\,dx \leqslant \frac{1}{16}\mu\int_{\mathbb{R}} u_{xxx}^2\,dx + Ce^{-lt}\int_{\mathbb{R}} \left(u_x^2 + u_{xx}^2\right)dx.
\tag{4.23}
$$

Similarly, we have from Lemma 2.2 with Cauchy–Schwartz inequality

$$
\begin{aligned}
&-2c_0\int_{\mathbb{R}} (\bar{\psi}v)_{xx}v_{xxx}\,dx \\
&\leqslant \frac{1}{16}\lambda\int_{\mathbb{R}} v_{xxx}^2\,dx + C\int_{\mathbb{R}} \left(\bar{\psi}_x^2v_x^2 + \bar{\psi}^2v_{xx}^2 + \bar{\psi}_{xx}^2v^2\right)dx \\
&\leqslant \frac{1}{16}\lambda\int_{\mathbb{R}} v_{xxx}^2\,dx + C\|\bar{\psi}_x\|_{L^\infty}^2\int_{\mathbb{R}} v_x^2\,dx + C\|\bar{\psi}\|_{L^\infty}^2\int_{\mathbb{R}} v_{xx}^2\,dx + C\|\bar{\psi}_{xx}\|_{L^\infty}^2\int_{\mathbb{R}} v^2\,dx \\
&\leqslant \frac{1}{16}\lambda\int_{\mathbb{R}} v_{xxx}^2\,dx + Ce^{-2(1-\alpha)t}\int_{\mathbb{R}} \left(v^2 + v_x^2 + v_{xx}^2\right)dx,
\end{aligned}
\tag{4.24}
$$

also

$$-2c_0\int_{\mathbb{R}}(\bar{\psi}u)_{xx}u_{xxx}\,dx \leqslant \frac{1}{16}\mu\int_{\mathbb{R}}u_{xxx}^2\,dx + Ce^{-2(1-\alpha)t}\int_{\mathbb{R}}\big(u^2+u_x^2+u_{xx}^2\big)\,dx, \tag{4.25}$$

and

$$-2c_0\int_{\mathbb{R}}(\bar{\theta}u)_{xx}v_{xxx}\,dx \leqslant \frac{1}{8}\lambda\int_{\mathbb{R}}v_{xxx}^2\,dx + Ce^{-2(1-\alpha)t}\int_{\mathbb{R}}\big(u^2+u_x^2+u_{xx}^2\big)\,dx. \tag{4.26}$$

Finally, we have by Lemma 2.2, Cauchy–Schwartz inequality and (3.4)

$$\begin{aligned}
&2c_0\int_{\mathbb{R}}F_{xx}(x,t)v_{xx}\,dx\\
&\quad\leqslant \delta e^{\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}}v_{xx}^2\,dx + \frac{c_0^2}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}}F_{xx}^2(x,t)\,dx\\
&\quad\leqslant \delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + \frac{C}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_{\mathbb{R}}\big(\bar{\psi}_{xxx}^2+\bar{\psi}_{xx}^2\bar{\theta}_x^2+\bar{\psi}_x^2\bar{\theta}_{xx}^2+\bar{\psi}^2\bar{\theta}_{xxx}^2+\bar{\theta}^2\bar{\psi}_{xxx}^2\big)\,dx\\
&\quad\leqslant \delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + \frac{C}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\delta^2e^{-2(1-\alpha)t}\\
&\quad\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}.
\end{aligned} \tag{4.27}$$

We get a similar estimate,

$$2\int_{\mathbb{R}}E_{xx}(x,t)v_{xx}\,dx \leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}. \tag{4.28}$$

Substituting (4.18)–(4.28) into (4.20), we get from (3.5)

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\big(u_{xx}^2+c_0v_{xx}^2\big)\,dx + (2-\varepsilon)(1-\alpha)\int_{\mathbb{R}}u_{xx}^2\,dx + \left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}\right\}\int_{\mathbb{R}}c_0v_{xx}^2\,dx\\
&\qquad+\frac{7}{8}(2-\varepsilon)\alpha\int_{\mathbb{R}}u_{xxx}^2\,dx + \frac{3}{4}\lambda\int_{\mathbb{R}}v_{xxx}^2\,dx\\
&\quad\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + Ce^{-2(1-\alpha)t}\int_{\mathbb{R}}\big(u^2+v^2+u_x^2+v_x^2+v_{xx}^2+u_{xx}^2\big)\,dx + Ce^{-lt}\int_{\mathbb{R}}\big(v_x^2+v_{xx}^2+u_{xx}^2\big)\,dx\\
&\quad\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + C(\delta+\delta_0)e^{-2(1-\alpha)t} + Ce^{-lt}\int_{\mathbb{R}}\big(v_x^2+v_{xx}^2+u_{xx}^2\big)\,dx.
\end{aligned} \tag{4.29}$$

Recalling the definition (4.2) of l, we obtain from (3.10) and (4.29)

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\big(u_{xx}^2+c_0v_{xx}^2\big)\,dx + l\int_{\mathbb{R}}\big(u_{xx}^2+c_0v_{xx}^2\big)\,dx\\
&\quad\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + C(\delta+\delta_0)e^{-2(1-\alpha)t} + Ce^{-lt}\int_{\mathbb{R}}\big(v_x^2+v_{xx}^2+u_{xx}^2\big)\,dx.
\end{aligned} \tag{4.30}$$

Noticing $\frac{l}{2} < 1-\alpha$, we easily deduce from (4.24) with the help of Lemma 4.1 and (3.5)

$$\int_{\mathbb{R}}\big(u_{xx}^2+c_0v_{xx}^2\big)\,dx \leqslant \Bigg(C\delta_0 + C\delta\int_0^t e^{-\{\frac{l}{2}+(1-\alpha)\}\tau}e^{l\tau}\,d\tau + C(\delta+\delta_0)\int_0^t e^{-2(1-\alpha)\tau}e^{l\tau}\,d\tau$$

$$+ C \int_0^t e^{-l\tau} \int_{\mathbb{R}} \left(v_x^2 + v_{xx}^2 + u_{xx}^2\right) dx\, e^{l\tau}\, d\tau \Biggr) e^{-lt}$$

$$\leqslant \left(C\delta_0 + C\delta + C(\delta + \delta_0) + C \int_0^t \int_{\mathbb{R}} \left(v_x^2 + v_{xx}^2 + u_{xx}^2\right) dx\, d\tau \right) e^{-lt}$$

$$\leqslant C(\delta + \delta_0) e^{-lt}. \tag{4.31}$$

The proof of Theorem 4.2 is completed by (4.16), (4.17) and (4.31). □

Finally, we verify the a priori assumption (4.1) is reasonable. Indeed, under this a priori assumption, we show (4.4) holds. Therefore, the assumption (4.1) is always true provided δ and δ_0 are sufficiently small.

5. Further discussion

We may also establish the L^p decay properties of the system (1.12), for the decay rates of the solution is dictated by the kernel function used for the integral form of the solution representation in Section 3.2. For related discussion and conclusion of the L^p decay properties of the solution, we refer to [3,16,17].

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